Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 5, pp. 739–744, 2002

NONLINEAR BENDING OF THIN ELASTIC RODS

Yu. V. Zakharov[1] **and K. G. Okhotkin**[2]

UDC 539.3

Exact solutions of the problem of nonlinear bending of thin rods under various fixing conditions and point dead loads are obtained. The solutions written in a unified parametric form and expressed in terms of the elliptic Jacobi functions are classified. These solutions depend on a single parameter — modulus of elliptic functions.

Introduction. An exact solution of the equation of nonlinear bending of a rod, written in the form of an equation of a nonlinear pendulum, was obtained first by Heinzerling [1] (see also [2]). Popov [3, 4] gave elliptic-integral solutions governing equilibrium of a rod, which depend on three parameters related implicitly to the boundary conditions and acting force, and studied possible configurations of the bent rod such as segments of the Euler elastica. Zakharov and Zakharenko [5] gave an exact solution for a rod under transverse loads, expressed in terms of the elliptic Jacobi functions, which depends on a single external parameter — modulus of elliptic functions related to the external force. Given this solution, one can determine shapes of the bent rod for an arbitrary number of inflection points. Levyakov [6] obtained exact solutions that describe equilibrium of a rod loaded by an axial force in terms of elliptic functions and studied the secondary loss of stability of the rod, which can occur under certain conditions.

In the present paper, we give an exact analytic solution of the problem of nonlinear bending of a rod loaded by a force whose direction remains unchanged during deformation of the rod. The critical loads and equilibrium curvilinear configurations of the loaded rod are calculated.

1. General Solution of the Problem of Rod Bending. We consider a thin inextensible rod of length L and flexural rigidity EI. The Cartesian coordinate system XOY is chosen in such a manner that the OX axis is directed along the undeformed straight rod, and the coordinate origin is located at its left end. The left end of the rod is fixed, and the right end is fixed or free. The rod is compressed by a dead force $\boldsymbol{P}$. We denote the arclength reckoned along the rod by l, the angle between a tangent at the current point to the rod axis and the OX axis by $\theta(l)$, and the Cartesian components of the force $\boldsymbol{P}$ by P_x and P_y.

We write the equation of equilibrium of the rod [4]

$$EI\,\frac{d^2\theta}{dl^2} - P_x\sin\theta + P_y\cos\theta = 0 \tag{1}$$

and reduce it to the equation of a nonlinear pendulum. We introduce the following notation: P is the magnitude of the dead load and φ_0 is the angle between the direction of the force P and the OX axis. To study all possible configurations of the rod, it suffices to consider the angle φ_0 varied from 0 to $\pi/2$. The angles $\varphi_0 > \pi/2$ correspond to rod extension and, hence, are not considered in the paper. The quantities P and φ_0 are assumed to be specified parameters of state. It is obvious that

$$P_x = -P\cos\varphi_0, \qquad P_y = P\sin\varphi_0. \tag{2}$$

Substituting (2) into Eq. (1), we obtain

$$\frac{d^2\theta}{dl^2} + \frac{P}{EI}\,\sin(\theta+\varphi_0) = 0.$$

[1]Kirenskii Institute of Physics, Siberian Division, Russian Academy of Sciences, Krasnoyarsk 660036. [2]Siberian State Technology University, Krasnoyarsk 660049. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 43, No. 5, pp. 124–131, September–October, 2002. Original article submitted April 29, 2002.

0021-8944/02/4305-0739 $27.00

We introduce the dimensionless length $t = l/L$, which varies from 0 to 1, and change the variable $\gamma = \theta + \varphi_0$. As a result, the last equation becomes

$$\frac{d^2\gamma}{dt^2} + q^2 \sin\gamma = 0, \tag{3}$$

where $q^2 = PL^2/(EI)$. The solution of Eq. (3) has the form

$$\gamma(t) = 2\arcsin\,[k\,\mathrm{sn}\,(qt + F_1, k)], \qquad \frac{d\gamma(t)}{dt} = 2kq\,\mathrm{cn}\,(qt + F_1, k), \tag{4}$$

where sn and cn are the elliptic Jacobi sine and cosine, respectively. The modulus of elliptic functions k and the parameter F_1 play the role of integration constants and they are related to the force P and angle φ_0 by boundary conditions for each case of rod bending.

We denote the argument of elliptic functions by

$$u = qt + F_1. \tag{5}$$

Using (4), we obtain

$$\cos\gamma(t) = 1 - 2k^2\,\mathrm{sn}^2\,u, \qquad \sin\gamma(t) = 2k\,\mathrm{sn}\,u\,\mathrm{dn}\,u, \tag{6}$$

where dn is the elliptic Jacobi delta function. Integration of the relations $dx/dl = \cos\theta$ and $dy/dl = \sin\theta$ yields the following expressions for the coordinates of an arbitrary point of the rod:

$$\begin{aligned} \frac{x}{L} &= \int_0^t \cos\theta\,dt = \int_0^t \cos\,(\gamma - \varphi_0)\,dt = \cos\varphi_0 \int_0^t \cos\gamma\,dt + \sin\varphi_0 \int_0^t \sin\gamma\,dt = X_0\cos\varphi_0 + Y_0\sin\varphi_0, \\ \frac{y}{L} &= \int_0^t \sin\theta\,dt = \int_0^t \sin\,(\gamma - \varphi_0)\,dt = \cos\varphi_0 \int_0^t \sin\gamma\,dt - \sin\varphi_0 \int_0^t \cos\gamma\,dt = Y_0\cos\varphi_0 - X_0\sin\varphi_0. \end{aligned} \tag{7}$$

Here

$$\begin{aligned} X_0 &= \int_0^t \cos\gamma\,dt = \int_0^t (1 - 2k^2\,\mathrm{sn}^2\,u)\,dt = -t + \frac{2}{q}\,[E(\mathrm{am}\,u) - E(\mathrm{am}\,F_1)], \\ Y_0 &= \int_0^t \sin\gamma\,dt = \int_0^t 2k\,\mathrm{sn}\,u\,\mathrm{dn}\,u\,dt = \frac{2k}{q}(\mathrm{cn}\,F_1 - \mathrm{cn}\,u), \end{aligned} \tag{8}$$

$E\,(\mathrm{am}\,u)$ is the incomplete elliptic integral of the second kind of the elliptic Jacobi amplitude. Expressions (7) and (8) determine configurations of the bent rod in the parametric form, and the role of the parameter is played by the reduced length of the rod t.

2. Bending of a Cantilevered Rod under a Dead Load. The boundary conditions have the form

$$\theta(0) = 0, \qquad \frac{d\theta(L)}{dl} = 0.$$

For Eq. (3), these conditions are written as

$$\gamma(0) = \varphi_0, \qquad \frac{d\gamma(1)}{dt} = 0. \tag{9}$$

With allowance for the first condition in (9), from (4) we obtain $\mathrm{sn}\,F_1 = \sin\,(\varphi_0/2)/k$ and, hence,

$$F_1 = F[\arcsin\,(\sin\,(\varphi_0/2)/k), k], \tag{10}$$

where F is the elliptic integral of the first kind and the modulus k varies within the limits $\sin\,(\varphi_0/2) < k < 1$.

The second condition in (9) implies $\mathrm{cn}\,(q + F_1) = 0$, which yields

$$q = (2n - 1)K(k) - F_1 \qquad (n = 1, 2, 3, \ldots). \tag{11}$$

Here $K(k)$ and $F(\varphi, k)$ are the complete and incomplete elliptic integrals of the first kind, respectively. Expressions (10) and (11) determine the spectrum of eigenvalues $q_n(k)$, which, in turn, determines the critical loads

$$P/P_{\mathrm{cr}} = (2/\pi)^2\{(2n - 1)K(k) - F[\arcsin\,(\sin\,(\varphi_0/2)/k), k]\}^2. \tag{12}$$

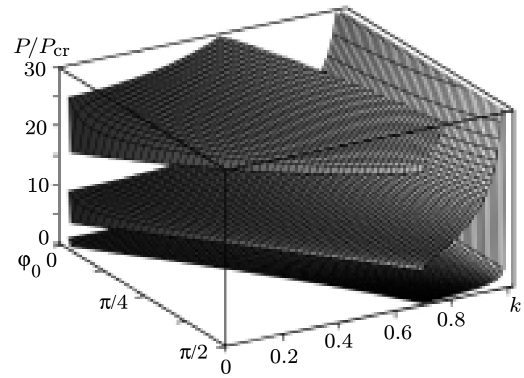

Fig. 1. Eigenvalue spectrum of the equation of a thin rod.

Here $P_{\rm cr} = (\pi/2)^2 EI/L^2$ is the Euler critical load and n enumerates the modes of the solution. Figure 1 shows the load $P/P_{\rm cr}$ versus the variables k and φ_0 in accordance with expression (12).

Configurations of the rod are determined by substituting q and F_1 from (10) and (11) into (7) and (8). Specifying the value of the external load P and mode number n, one obtains a certain configuration of the rod determined by a single parameter — modulus k related to the force P by relation (12). The minimum value of $k = \sin(\varphi_0/2)$ corresponds to the minimum critical force. As $P \to \infty$, $k \to 1$.

Setting $n = 1$, we obtain the Euler critical force. Lavrent'ev and Ishlinskii [7] called the critical loads corresponding to $n > 1$ the dynamic thresholds of buckling. These loads can be reached in the case of shock loading where the rise time of the pulse is shorter than the relaxation time of the system.

Usually, the critical loads are determined by solving a linear equation of equilibrium (see [8]). For an elastic cantilevered rod loaded by an axial ($\varphi_0 = 0$) force P, the critical loads $P_n = (2n-1)^2(\pi/2)^2\alpha_r/L^2$ $(n = 1, 2, \ldots)$ are the eigenvalues of the boundary-value problem

$$y'' + (P/\alpha_r)y = 0, \qquad y(0) = 0, \quad y'(L) = 0,$$

where $\alpha_r = EI$ is the flexural rigidity and $y(x)$ is the deflection of the rod.

3. Bending of a Cantilevered Rod under a Transverse Load. For a transverse load, we have $\varphi_0 = \pi/2$. In this case, the eigenvalue spectrum (12) has the form

$$P/P_{\rm cr} = (2/\pi)^2\{(2n-1)K(k) - F\,[\arcsin(\sqrt{2}/(2k)), k]\}^2 \quad (n = 1, 2, 3, \ldots). \tag{13}$$

According to [5], we consider the case $n = 1$. If $P = 0$, then $k^2 = 1/2$. As $P \to \infty$, $k^2 \to 1$. If the force acts in the opposite direction ($P \to -\infty$), one should replace k by k' in all the expressions, where the additional modulus k' is determined from the relation $k^2 + k'^2 = 1$. For $P = 0$, we have $k'^2 = 1/2$ and as $P \to -\infty$, $k'^2 \to 1$. This implies that the rod is deflected from the equilibrium position under any nonzero force P. In this case, if the load increases monotonically, no critical values exist.

If $n > 1$, it follows from (13) that critical loads exist for $k^2 = 1/2$:

$$P_n = (n-1)^2[(4/\pi)K(\sqrt{2}/2)]^2 P_{\rm cr} \approx 5.6(n-1)^2 P_{\rm cr} \qquad (n = 1, 2, \ldots).$$

For $n = 1$, no critical loads exist and for $n > 1$, the critical loads are dynamic.

For $\gamma = \theta + \pi/2$, the solution of Eq. (3) becomes

$$\gamma(t) = 2\arcsin(k\,\mathrm{sn}\,((pK(k) - F_1)t + F_1, k)),$$

$$\cos\gamma(t) = 1 - 2k^2\,\mathrm{sn}^2\,u, \qquad \sin\gamma(t) = 2k\,\mathrm{sn}\,u\,\mathrm{dn}\,u,$$

where $p = 2n - 1 = 1, 3, 5, \ldots$, $u = (pK - F_1)t + F_1$ is the argument, and $F_1 \equiv F\,[\arcsin(\sqrt{2}/(2k)), k]$.

Equations (7) and (8) determining the coordinates x and y of an arbitrary point of the rod have the form

$$\frac{x}{L} = \frac{2k}{pK - F_1}\Big[\Big(1 - \frac{1}{(2k)^2}\Big)^{1/2} - \mathrm{cn}\,u\Big], \qquad \frac{y}{L} = t - \frac{2}{pK - F_1}\,[E(\mathrm{am}\,u, k) - E_1]. \tag{14}$$

Here $E(\varphi, k)$ is the incomplete elliptic integral of the second kind and $E\,[\arcsin(\sqrt{2}/(2k)), k] \equiv E_1$. Equation (14) was obtained using the relation $\mathrm{cn}\,(F_1, k) = \cos(\arcsin(\sqrt{2}/(2k))) = (1 - 1/(2k)^2)^{1/2}$.

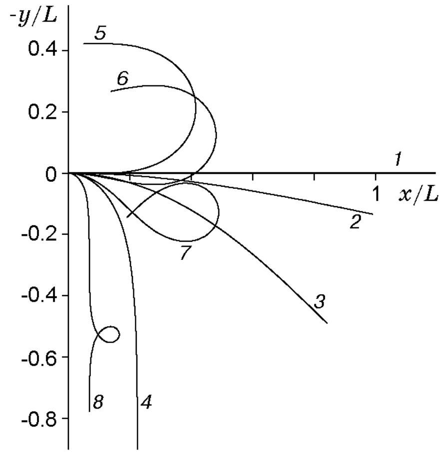

Fig. 2. Equilibrium configurations of the cantilevered rod under a transverse load: curves 1–4 refer to the static mode ($n = 1$) and curves 5–8 refer to the first dynamic mode ($n = 2$) for $k^2 = 0.5$ (1 and 5), 0.6 (2 and 6), 0.85 (3 and 7), and $1 - 10^{-5}$ (4 and 8).

Equations (14) determine the shape of a cantilever bent strongly by static ($n = 1$) and dynamic ($n = 2, 3, \ldots$) loads in the parametric form ($0 \leqslant t \leqslant 1$). The modulus k related to the force P varies within the interval $1/2 \leqslant k^2 \leqslant 1$ and characterizes the curvature of the cantilever. For each critical load, the dependence $k^2(P/P_{\rm cr})$ is determined by Eq. (13) for a corresponding value of $p = 2n - 1$.

Figure 2 shows the configurations of the bent rod for static ($p = 1$) and dynamic ($p = 3$) modes for different values of the load applied (see [5]).

For dynamic modes, the result obtained is paradoxical: under the action of a pulsed load whose rise time is shorter than the relaxation time of the system, the rod is deflected toward the acting force. Physical nature of this metastable state is not understood. Zakharov and Zakharenko [5] considered some cases where similar effects are manifested. It is noteworthy that Shkutin [9] also obtained similar results by numerical methods.

4. Bending of a Cantilevered Rod under Axial Compression. In the case of an axial load, we have $\varphi_0 = 0$. The eigenvalue spectrum (12) takes the form

$$P/P_{\rm cr} = (2/\pi)^2[(2n-1)K(k)]^2 \qquad (n = 1, 2, 3, \ldots). \tag{15}$$

The equation describing rod bending is obtained by setting $\gamma = \theta$ in (3). The solution of this equation has the form

$$\theta(t) = 2\arcsin\,[k\,{\rm sn}\,(pK(k)t, k)], \quad \cos\theta(t) = 1 - 2k^2\,{\rm sn}^2\,u, \quad \sin\theta(t) = 2k\,{\rm sn}\,u\,{\rm dn}\,u,$$

where $u = pKt$ and $p = 2n - 1 = 1, 3, 5, \ldots$.

Expressions (7) and (8) for the x and y coordinates of an arbitrary point of the rod become

$$x/L = -t + 2E\,({\rm am}\,u)/(pK), \qquad y/L = 2k\,(1 - {\rm cn}\,u)/(pK).$$

These formulas are obtained with allowance for $F_1 = 0$ for the axial force. Figure 3 shows the configurations of the rod for static ($p = 1$) and dynamic ($p = 3$) modes for different values of the load applied.

5. Bending of a Rod with Clamped Ends under Axial Compression. In the case of axial compression ($P_x = -P$ and $P_y = 0$), one should set $\gamma = \theta$ in Eq. (3) and solution (4)–(6). The bent axis of the rod is determined by expressions (7) and (8) for $\varphi_0 = 0$. The boundary conditions have the form

$$\theta(0) = 0, \qquad \theta(1) = 0. \tag{16}$$

Moreover, we consider the additional relation

$$\frac{y(L)}{L} = \int\limits_0^1 \sin\theta\,dt = \frac{2k}{q}\,[{\rm cn}\,F_1 - {\rm cn}\,(q + F_1)] = 0, \tag{17}$$

which is the condition of zero ordinates of the clamped ends of the rod.

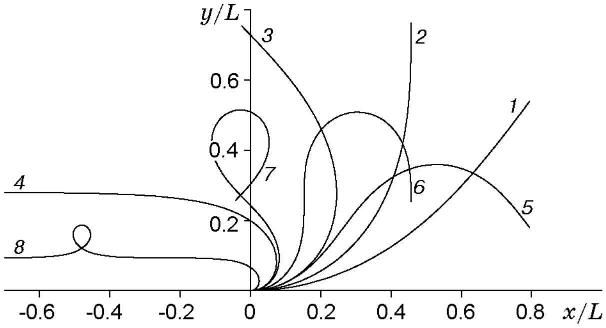

Fig. 3

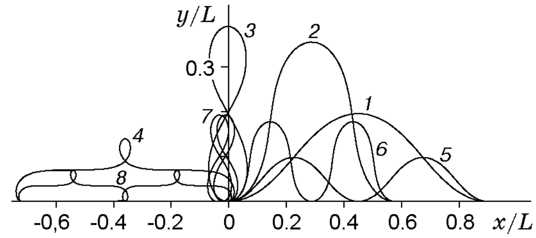

Fig. 4

Fig. 3. Equilibrium configurations of a cantilevered rod under axial compression: curves 1–4 refer to the static mode ($n = 1$) and curves 5–8 refer to the first dynamic mode ($n = 2$) for $k^2 = 0.2$ (1 and 5), 0.5 (2 and 6), 0.85 (3 and 7), and $1 - 10^{-5}$ (4 and 8).

Fig. 4. Equilibrium configurations of a cantilevered rod under axial compression: curves 1–4 refer to the static mode ($n = 1$) and curves 5–8 refer to the first dynamic mode ($n = 2$) for $k^2 = 0.1$ (1 and 5), 0.4 (2 and 6), 0.85 (3 and 7), and $1 - 10^{-5}$ (4 and 8).

Taking into account (16) and the properties of zeros of elliptic sine,

$$\mathrm{sn}\,(0 + F_1, k) = 0, \quad F_1 = 2n_1K(k), \qquad \mathrm{sn}\,(q + F_1, k) = 0, \quad q + F_1 = 2n_2K(k),$$

from (4) we obtain $q = 2n_2K(k) - F_1 = 2(n_2 - n_1)K(k)$ or $q = 2mK(k)$, $F_1 = 2K(k)$. Here $(n_1, n_2, m) = 1, 2, 3, \ldots$. By virtue of (17), the eigenvalues are given more rigorously by $q = 4nK(k)$. The eigenvalue spectrum (critical loads) have the form

$$P/P_{\mathrm{cr}} = (2/\pi)^2[4nK(k)]^2 \qquad (n = 1, 2, 3, \ldots).$$

Expressions (7) and (8) determining configurations of the rod become

$$x/L = -t + 2E\,(\mathrm{am}\,u)/(4nK), \quad y/L = 2k(1 - \mathrm{cn}\,u)/(4nK), \quad u = 4nK(k)t.$$

These formulas are obtained with allowance for periodicity of elliptic functions. Figure 4 shows the configurations of the rod for the first two modes.

6. Bending of a Rod with Hinged Ends under Axial Compression. For axial compression ($P_x = -P$ and $P_y = 0$), we set $\gamma = \theta$ in Eq. (3) and solution (4)–(6). The configurations of the bent rod is determined by (7) and (8) for $\varphi_0 = 0$. The boundary conditions for hinged ends have the form

$$\frac{d\theta(0)}{dl} = 0, \qquad \frac{d\theta(1)}{dl} = 0. \tag{18}$$

Moreover, we use condition (17), which requires that the ordinates of the rod ends are equal to zero.

Taking into account (18) and the properties of zeros of elliptic cosine

$$\mathrm{cn}\,(0 + F_1, k) = 0, \quad F_1 = (2n_1 + 1)K(k), \qquad \mathrm{cn}\,(q + F_1, k) = 0, \quad q + F_1 = (2n_2 + 1)K(k),$$

from (4) we obtain $q = (2n_2 + 1)K(k) - F_1 = 2(n_2 - n_1)K(k)$ or $q = 2nK(k)$ and $F_1 = K(k)$. Here $(n_1, n_2, n) = 1, 2, 3, \ldots$. The additional condition (17) is satisfied identically. The eigenvalue spectrum (critical loads) is given by

$$P/P_{\mathrm{cr}} = (2/\pi)^2[2nK(k)]^2 \qquad (n = 1, 2, 3, \ldots).$$

Expressions (7) and (8) for the rod coordinates have the form

$$x/L = -t + 2[E\,(\mathrm{am}\,u) - E\,(k)]/(2nK), \quad y/L = -2k\,\mathrm{cn}\,u/(2nK), \quad u = 2nK(k)t + K(k).$$

These formulas are written with allowance for periodicity of elliptic functions and the relation $E\,(\mathrm{am}\,((2n - 1)K, k), k) = E((2n - 1)\pi/2, k) = 2nE(k) + E(\pi/2, k) = (2n - 1)E$. Figure 5 shows the configurations of the rod for first two modes.

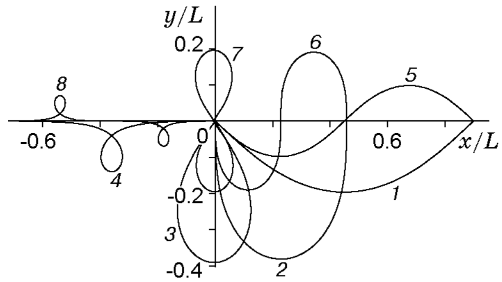

Fig. 5. Equilibrium configurations of a rod with hinged ends under axial compression: curves 1–4 refer to the static mode ($n = 1$) and curves 5–8 refer to the first dynamic mode ($n = 2$) for $k^2 = 0.1$ (1 and 5), 0.5 (2 and 6), 0.83 (3 and 7), and $1 - 10^{-5}$ (4 and 8).

Conclusions. The results obtained above can be used to verify numerical methods of solving nonlinear equations of bending of rods [9].

Zakharov [10] considered the magnetic reversal of a magnetic system in a layer across anisotropy and showed that, after the dynamic threshold, magnetization in the layer turns from the equilibrium position and becomes opposite to the field over the entire thickness of the layer in a similar manner as a rod loaded by a transverse force is bent oppositely to the force direction after the first dynamic threshold is reached. These effects can be observed in tests on rods under shock loading. The problem of magnetization reversal along and across the direction of magnetic anisotropy of a magnetic layer with nonsymmetric boundary conditions is similar to the Euler problem of stability of an elastic rod. It should be noted that magnetic systems are more convenient to study the dynamic buckling since many experiments on these systems can easily be performed. For a magnetic system, modes of magnetization reversal are similar to axial and transverse loads applied to the free end of an elastic rod whose other end is clamped, which stimulated the analysis of nonlinear solutions for bending of a thin rod.

The authors are grateful to K. S. Alexandrov, L. I. Shkutin, S. G. Ovchinnikov, and V. G. Sukhovol'skii for their attention to the work.

This work was supported by the Russian Foundation for Fundamental Research (Grant Nos. 02-01-01017 and 02-01-06308).

REFERENCES

1. H. Heinzerling, "Mathematische Behandlung einiger grundlegender Fragen des Knicksproblems des geraden Stabes," Diss. Karlsruhe (1938).
2. L. Collatz, *Eigenvalue Problems* (*with Engineering Applications*) [Russian translation], Fizmatgiz, Moscow (1968).
3. E. P. Popov, *Nonlinear Static Problems of Thin Rods* [in Russian], OGIZ, Moscow–Leningrad (1948).
4. E. P. Popov, *Theory and Calculation of Flexible Elastic Rods* [in Russian], Nauka, Moscow (1986).
5. Yu. V. Zakharov and A. A. Zakharenko, "Dynamic instability in the nonlinear problem of a cantilever," *Vychisl. Tekhnol.*, **4**, No. 1, 48–54 (1999).
6. S. V. Levyakov, "States of equilibrium and secondary loss of stability of a straight rod loaded by an axial force," *J. Appl. Mech. Tech. Phys.*, **42**, No. 2, 321–327 (2001).
7. M. A. Lavrent'ev and A. Yu. Ishlinskii, "Dynamic buckling modes of elastic systems," *Dokl. Akad. Nauk SSSR*, **64**, No. 6, 779–782 (1949).
8. A. S. Vol'mir, *Stability of Deformable Systems* [in Russian], Nauka, Moscow (1967).
9. L. I. Shkutin, "Numerical analysis of the branched forms of bending for a rod," *J. Appl. Mech. Tech. Phys.*, **42**, No. 2, 310–315 (2001).
10. Yu. V. Zakharov, "Static and dynamic instability of a ferromagnetic layer under magnetic reversal," *Dokl. Ross. Akad. Nauk*, **344**, No. 3, 328–332 (1995).